Traffic of particles in complex networks

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The study of the information flow through communication networks, such as the Internet, is of great importance. In the Internet, information flows in discrete units ("packets"), and the capacity of storage and processing of information of computers is finite. Thus if there are many packets walking on the network at the same time, they will interfere with each other. To understand this, we propose an idealized model, in which many particles move randomly on the network, and the nodes support limited numbers of particles. The maximum number of packets supported by a node can be any positive integer, and can be different for each node. We analyze the statistical properties of this model, obtaining analytical expressions for the mean occupation of each node, for different network topologies. The analytical results are shown to be in agreement with numerical simulations.

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I. INTRODUCTION

In the last years the study of dynamical processes in complex networks has developed rapidly, driven by applications of concepts of network science in many areas, including physics, sociology, biology, and others [1–3]. In particular, the dynamics of information flow in various kinds of communication networks has been much studied [4–9]. One particularly important communication network is the Internet. In the Internet, communication between computers is achieved by means of the exchange of small discrete units of data, the "packets." The information flow can thus be understood in terms of the traffic of particles moving through a network. Due to the limited capacity of processing and storage of each computer, for high traffic the particles will interfere with each other's motions. The study of this kind of collective dynamics poses a challenging problem to statistical physics. One way to try to quantify the effects of high traffic and congestion is by means of centrality measures such as the *betweenness centrality*, which are purely topological quantities [10–15]. Using this idea, cascades of failures resulting from the failure of a single node have also been extensively studied [16–20]. Although topological measures like the betweenness centrality are very useful in the study of information transport in networks, they offer only an incomplete characterization of the dynamics. Models of the full particle hopping dynamics have been proposed and investigated [21–31]. However, these models are in most cases too complicated to be solved analytically.

In this paper, we propose a simple model of this dynamical process which embodies many of its main properties, but which is simple enough as to allow us to obtain some analytical results. Our point of view is that even though this model is very idealized, it is, nevertheless, useful in uncovering some of the basic dynamical features of the collective behavior of the packets in the Internet. In our model, particles move in a graph (or network). Each node of the graph represents a computer in the Internet, and the links mirror the corresponding connections among the computers. At any time each particle (or packet) is located at one of the nodes of the network. The finite capacity of the computers to process the packets is modeled by assigning to each node i a maximum number of particles m_i it can support, with $m_i \geq 1$. m_i can be different for each node i, taking into account the fact that different computers may have different capacities. Packet motion is modeled by prescribing that the particles perform a random walk in the network, subject to the constraint that no particle can move to a node which already has its maximum allowed number of particles. This constraint introduces an effective interparticle interaction which is responsible for the collective behavior mentioned above.

A particular case of this model was proposed and solved in a previous paper [32]. In that model, all nodes were constrained to be occupied by at most one particle. It was shown that this made the statistics of the system equivalent to that of a Fermi gas, which allowed the grand partition function of the system to be computed analytically, and thus all the statistical properties were found. The model of Ref. [32] corresponds to $m_i=1$ for all i. In this paper, we propose and study an extensive generalization of that model, letting m_i be greater than 1, and allowing it to assume different values for different nodes. This complicates the calculation of the partition function, but we find that by using a mathematical trick we can find an analytical expression for it. Using this, we find expressions for the average occupation number of each node, thus completely characterizing the statistical properties of the system. Simulations of the constrained particle random walk confirm the accuracy of the analytical results, which have no adjustable parameters of any kind. The theory is tested for Erdös-Rényi and Barabási-Álbert networks, and the theoretical predictions are confirmed by the simulations in every case.

This paper is organized as follows. We first briefly review in Sec. II the results of Ref. [32] which will be used here. We then generalize the model in Sec. III for arbitrary m_i. We

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discuss the conceptual and technical problems brought by the *distinguishability* of the particles, and how they can be overcome by a mathematical trick which allows us to sum the grand-partition function. We use this general result in Sec. IV to study the particular case of homogeneous capacity, in which the m_i are the same for all i. An analytical expression for the occupation number of each node is found for this case, and tested by simulation in various network topologies. In Sec. V we tackle the general case of arbitrary m_i depending on i (inhomogeneous capacities), and we use the expression for the occupation number thus obtained to study the interesting case of m_i proportional to the degree of the node i. Simulations are again used to test the theoretical predictions, which are found to be fully verified to within numerical accuracy. We close in Sec. VI with some remarks and conclusions.

II. THE CASE $m_i=1$: THE FERMI-DIRAC DISTRIBUTION

We start with a quick review of the results of Ref. [32] which are relevant to this work. It concerns the case where $m_i=1$ for all nodes i. In other words, all nodes can support at most one particle, which is the simplest nontrivial case of the general model.

It is argued in Ref. [32] that the constraint that no two particles can occupy the same node works just like the exclusion principle for identical fermions in quantum mechanics. The particles are then analogous to free fermions, and the nodes are the equivalent of single-particle states of a quantum system. The links can be thought of as selection rules for transitions among the states. From this correspondence between the particle dynamics in the network and a noninteracting Fermi-Dirac gas, we conclude that the average occupation number of node i is given by the Fermi-Dirac distribution:

$$\langle n_i \rangle = \frac{1}{e^{\alpha+\beta\varepsilon_i}+1}, \tag{1}$$

where ε_i is the energy associated with the ith node, and α is related to the chemical potential. ε_i is determined by the probability that a single isolated particle performing a random walk in the network is found in node i. The result is (see Ref. [32] for more details)

$$\beta\varepsilon_i = \ln(Ck_i^{-1}), \tag{2}$$

where k_i is the degree (the number of neighboring nodes) of node i, and C is a constant. This equation relates the energy of a node to a simple local property of the node, namely the number of neighbors it is connected to. The "chemical potential" α is determined by prescribing the average number of particles N present on the network:

$$\sum_i \langle n_i \rangle = N, \tag{3}$$

where the sum is over all nodes. Since this is a grand-canonical ensemble, the actual number of particles fluctuates around N, but for large N these fluctuations are very small.

Equations (1)–(3) allow one to calculate the average occupation number of any node, for a given network and for a prescribed (average) number of particles. Simulation results presented in Ref. [32] confirm the accuracy of these predictions. The generalization of this approach to arbitrary m_i poses new conceptual and technical problems, to which we now turn our attention.

III. THE PARTITION FUNCTION AND THE ISSUE OF DISTINGUISHABILITY

Now we generalize from the situation in which at most one particle can be located at a node to the case in which a maximum of m_i particles can occupy node i. One might be tempted to assume that the average occupation $\langle n_i \rangle$ should be given by the so-called *Gentile distribution* [33], which describes hypothetical identical particles whose statistical properties are in between fermions and bosons. These *gentilions* are parametrized by an integer l, and they have the property that only at most l particles can occupy a quantum state. Their average occupation number is given by

$$\langle n_i \rangle = \frac{1 + le^{-\beta(\varepsilon_i-\mu)(l+1)} - (l+1)e^{-\beta(\varepsilon_i-\mu)l}}{e^{\beta(\varepsilon_i-\mu)}(1-e^{-\beta(\varepsilon_i-\mu)})(1-e^{-\beta(\varepsilon_i-\mu)(l+1)})}. \tag{4}$$

The Gentile distribution may at first seem the natural generalization to Eq. (1), and the solution to our problem. That this is not true can be seen immediately by observing that the limit $l\to\infty$ of Eq. (4) is the Bose-Einstein distribution, and not the Boltzmann distribution, which is the correct one to describe packets moving independently in the network.

The problem lies in the fact that our particles are classical entities, and therefore they are *distinguishable* from each other, whereas the Gentile (and the Bose-Einstein) distribution assumes they are indistinguishable. In order to find the correct expression for $\langle n_i \rangle$, we must take their distinguishability into account.

Let us denote by $\{n_i\}=\{n_1,n_2,\ldots\}$ a state of the system with n_1 particles in node 1, n_2 particles in node 2, etc. The energy associated with this configuration is $\Sigma_i \varepsilon_i n_i$, with the single-state energies ε_i being given by Eq. (2). The canonical partition function Z_N corresponding to exactly N distinguishable particles being in the network is

$$Z_N = \sum_{\{n_i\}}{}' \frac{N!}{n_1!\,n_2!\cdots n_i!\cdots} e^{-\beta\Sigma_i \varepsilon_i n_i}, \tag{5}$$

where the multinomial factor $n_1!n_2!\cdots$ comes from the fact that the particles are distinguishable. The symbol Σ' indicates that the indices n_i in the sum have to satisfy the constraint $\Sigma_i n_i=N$, besides the constraint resulting from the finite capacity of the nodes, $n_i \le m_i$. These two conditions make it impracticable to perform the sum. We thus go to the grand-canonical ensemble, a usual expedient in this kind of situation. The grand-canonical partition function Ξ is defined by

$$\Xi = \sum_{N=1}^{\infty} e^{\mu N} Z_N. \tag{6}$$

For technical reasons which will become clear in a moment, we choose to work with the rescaled canonical partition functions $\bar{Z}_N$, given by

$$\bar{Z}_N = \frac{Z_N}{N!} = \sum_{\{n_i\}}{}' \frac{1}{n_1!\, n_2!\, \cdots n_i!\, \cdots} e^{-\beta \Sigma_i \varepsilon_i n_i}. \tag{7}$$

Since the averages of any measurable quantity are calculated in the canonical ensemble from derivatives of the logarithm of the partition function, the predictions for the average occupation number $\langle n_i \rangle$ are exactly the same using either Z_N or $\bar{Z}_N$. The only issue is that $\bar{Z}_N$ has problems with extensivity (an unusual version of Gibbs' paradox). This will not trouble us, however, because all we are interested in are average quantities like $\langle n_i \rangle$.

When we go to the grand-canonical ensemble, we can use $\bar{Z}_N$ in Eq. (6) instead of Z_N, and the new grand partition function will give the same results for average quantities, for a large enough average particle number. We shall for simplicity denote this grand partition function by the same symbol Ξ we used for the previous one. No confusion should ensue from this, as from now on this is the only grand partition function we shall use. From Eqs. (6) and (7), it is given by

$$\Xi = \sum_{N=1}^{\infty} \sum_{\{n_i\}}{}' \frac{e^{-\beta \Sigma_i (\varepsilon_i - \mu) n_i}}{n_1!\, n_2!\, \cdots}. \tag{8}$$

The sums appearing in the above expression can now be reordered and expressed as independent sums in the n_i, without the global constraint $\Sigma_i n_i = N$:

$$\Xi = \sum_{n_1=0}^{m_1} \sum_{n_2=0}^{m_2} \cdots \frac{1}{n_1!\, n_2!\, \cdots} e^{-\beta(\varepsilon_1 - \mu) n_1 - \beta(\varepsilon_2 - \mu) n_2 \cdots}. \tag{9}$$

This is the reason we use the grand canonical ensemble, as now this expression is manageable. In fact, inspection of Eq. (9) reveals that the sums can be factorized:

$$\Xi = \left(\sum_{n_1=0}^{m_1} \frac{1}{n_1!} e^{-\beta(\varepsilon_1 - \mu) n_1} \right) \cdots \left(\sum_{n_j=0}^{m_j} \frac{1}{n_j!} e^{-\beta(\varepsilon_j - \mu) n_j} \right) \cdots . \tag{10}$$

This factorization was only possible because we used $\bar{Z}_N$ in the construction of the grand partition function, which got rid of the $N!$ factors. This explains our choice of using $\bar{Z}_N$ instead of Z_N. We again emphasize that this choice does not affect the physical predictions of the model, and this will be confirmed by the extensive numerical simulations presented in the following sections.

From the grand-canonical partition function, we calculate the average number of particles in state i by

$$\langle n_i \rangle = -\frac{1}{\beta} \frac{1}{\Xi} \frac{\partial \Xi}{\partial \varepsilon_i} = -\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \varepsilon_i}. \tag{11}$$

Using expression (10) for Ξ in the above equation, we get

$$\langle n_i \rangle = \frac{\sum_{n=1}^{m_i} \frac{1}{(n-1)!} e^{-\beta(\varepsilon_i - \mu) n}}{\sum_{n=0}^{m_i} \frac{1}{n!} e^{-\beta(\varepsilon_i - \mu) n}}. \tag{12}$$

Finally, using Eq. (2) to write the energy of a node i in terms of its degree k_i, we have

$$\langle n_i \rangle = \frac{\sum_{n=1}^{m_i} \frac{1}{(n-1)!} (A k_i)^n}{\sum_{n=0}^{m_i} \frac{1}{n!} (A k_i)^n}, \tag{13}$$

where $A = B e^{\mu}$ (B=const) contains the contribution of the chemical potential. This expression has the property that two nodes with the same degree have the same average occupation. We also note that $\langle n_i \rangle \to m_i$ as $k_i \to \infty$, so the highly connected nodes (hubs) tend to become fully occupied, which is expected. We observe that for the case of $m_i = 1$ for all i, the above equation reduces to Eq. (1), and so the Fermi-Dirac distribution of Ref. [32] is recovered as a particular case.

The chemical potential μ, or equivalently A, is determined by prescribing the average total number of particles N present in the network:

$$-\frac{1}{\beta} \frac{\partial \ln \Xi}{\partial \mu} = \sum_i \langle n_i \rangle = N. \tag{14}$$

This equation must be solved for μ, given N.

IV. NETWORKS WITH HOMOGENEOUS CAPACITY (m_i=const)

We first test the results of the last section in the case of a homogeneous capacity network, in which all nodes can support the same number of particles m. This means that $m_i = m$=const for all i. This assumption allows us to write the equation that determines the chemical potential, Eq. (14), in terms of the degree distribution p_k of the network. Indeed, if p_k is the probability of a node having degree k, we can rewrite the sum as a sum over the degrees, and we get the following equation:

$$\sum_{k=1}^{\infty} p_k \frac{\sum_{n=1}^{m} \frac{1}{(n-1)!} (Ak)^n}{\sum_{n=0}^{m} \frac{1}{n!} (Ak)^n} = \rho, \tag{15}$$

where $\rho = N/M$ is the ratio of the number of particles N to the number of nodes M—in other words, the particle density. Given the degree distribution and the particle density, A is determined by solving numerically Eq. (15), and its value is then used in Eq. (13) to find $\langle n_i \rangle$.

We want to test the predictions of Eqs. (13) and (15) by means of computer simulations. As described in Sec. I, the particles perform a random walk on the network, restricted by the condition that they cannot go to a node that already has its maximum number of occupying particles, in this case

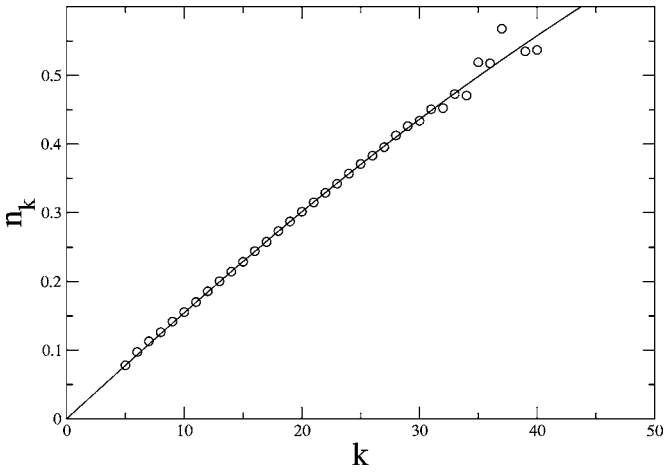

FIG. 1. Mean occupation $\langle n_k \rangle$ of nodes with degree k, for $m_i = m = 2$, in an Erdös-Rényi random network with 10^4 nodes, with $\rho = 0.3$. Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (16). The value of A in Eq. (16) is found from Eq. (15) to be $A = 64.02$.

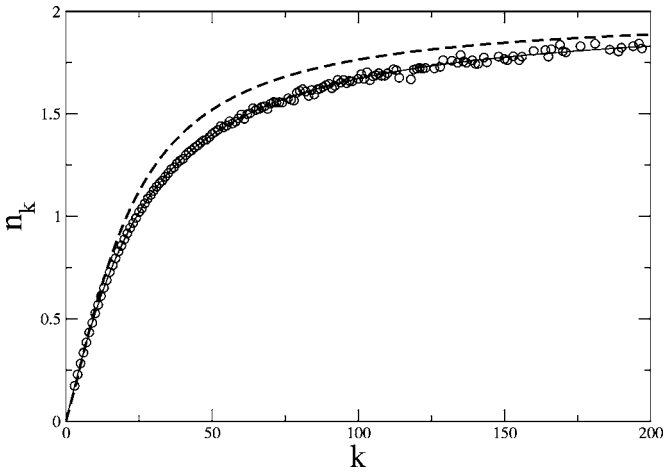

FIG. 2. Mean occupation $\langle n_k \rangle$ of nodes with degree k, for $m_i = m = 2$, in a Barabási-Álbert scale-free network with 10^5 nodes, with $\rho = 0.3$. Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (16). The value of A in Eq. (16) is found from Eq. (15) to be $A = 17.15$. The dashed line is the prediction of the Gentile statistics.

m. The algorithm we use to simulate this stochastic process is as follows. Each time step, a particle is randomly selected with uniform probability. One of the neighboring nodes of the node the particle is currently occupying is randomly chosen, also with uniform probability among the neighbors. If the chosen node can support the additional particle, it moves there; if the node is occupied to the full capacity, however, the move fails, and the particle stays where it is. The next time step starts, another particle is selected, and so on.

A number $N \gg 1$ of particles is distributed through the network's nodes initially in a random way, and then the stochastic process described above is iterated a large number of times, so that the system reaches equilibrium. Then the process is iterated further, and the average occupancy of the nodes is recorded. This gives us a numerical measure of $\langle n_i \rangle$, which we compare with the theoretical prediction of Eq. (13).

We use two networks to do the simulations: a random Erdös-Rényi network with 10^4 nodes and 10^5 edges [34]; and a scale-free network grown using the Barabási-Álbert algorithm [35–42], with 10^5 nodes, and an average of six links per node. We put $N = 0.3M$ particles in all simulations, that is, $\rho = 0.3$.

We start with the case $m = 2$. In this case each node can support at most two particles. The mean occupation of each node is given by Eq. (13) with $m = 2$:

$$\langle n_i \rangle = \frac{Ak_i + (Ak_i)^2}{1 + Ak_i + \frac{1}{2}(Ak_i)^2}. \tag{16}$$

The results of simulation are shown in Figs. 1 and 2.

We see that the theory matches very well the results of the simulation, for both networks. We note that the Gentile statistics predicts a different result [see Eq. (4)], shown in Fig. 2 as the dashed line. It is clear that the Gentile statistics does not match the simulation; this is a consequence of the distinguishability of the particles.

Figures 3 and 4 display the results for $m = 3$. We see again a very good agreement between simulation and theory. The prediction of the Gentile statistics is again clearly wrong. We have simulated the particle transport in homogeneous capacity networks for some other values of m, and also for some other network topologies, and we have always found that Eq. (13) is in almost perfect agreement with the simulations in all cases.

V. NETWORKS WITH INHOMOGENEOUS CAPACITY

In the previous section we assumed that the capacity m_i of all nodes was the same. In real communication networks,

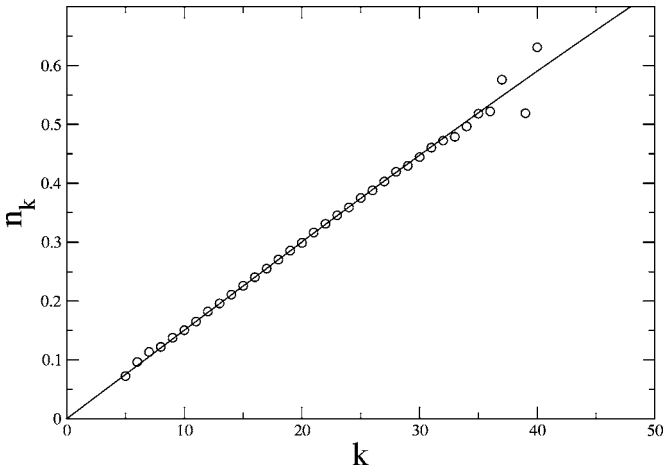

FIG. 3. Mean occupation $\langle n_k \rangle$ of nodes with degree k, for $m_i = m = 3$, in an Erdös-Rényi random network with 10^4 nodes, with $\rho = 0.3$. Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (13).

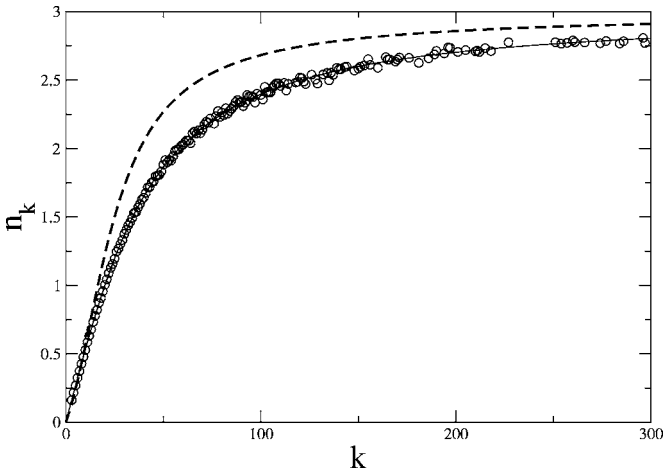

FIG. 4. Mean occupation $\langle n_k \rangle$ of nodes with degree k, for $m_i = m = 3$, in a Barabási-Álbert scale-free network with 10^5 nodes, with $\rho = 0.3$. Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (13). The dashed line is the prediction of the Gentile statistics.

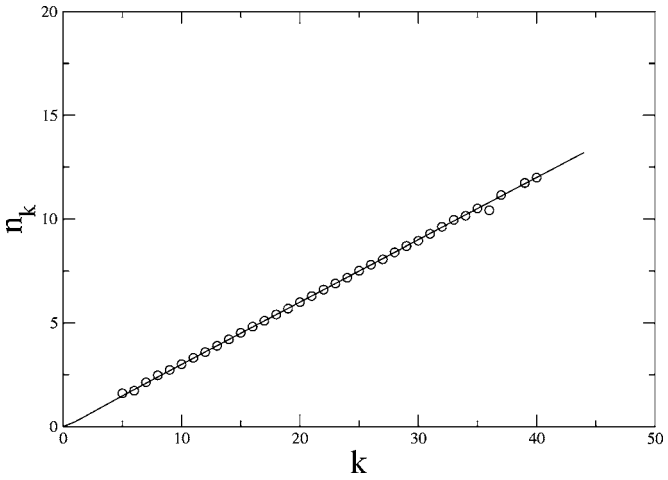

FIG. 5. Mean occupation n_k of nodes with degree k, with each node i supporting at most k_i particles, in a random Erdös-Rényi network with the same parameters as in Fig. 1. The number of particles is 30% of the maximum number supported by the network (which is the sum of the degrees of all nodes). Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (18).

however, the processing power of the nodes is distributed in a highly heterogeneous way. In the Internet, for example, some computers are much more powerful than others, and are thus able to forward packets at a much higher rate. In this section we apply the general results of Sec. III to the case of nonconstant m_i.

The results of the previous section indicate that highly connected nodes (*hubs*) have to deal with a high amount of traffic. It would therefore make sense to have computers with high capacity assigned to the hubs. We thus assume that the capacity m_i of a node depends only on its degree, $m_i = m(k_i)$, with $m(k_i)$ being an increasing function of k_i. For definiteness, we take

$$m(k_i) = k_i. \tag{17}$$

The number of supported particles of a node is thus proportional to its number of connections. Using this expression for m_i, we can after some elementary manipulations of Eq. (13) write the average occupation $\langle n_i \rangle$ as

$$\langle n_i \rangle = Bk_i - \frac{(Bk_i)^{k_i+1}}{k_i!} \frac{1}{\sum_{n=0}^{k_i} \frac{1}{n!}(Bk_i)^n}. \tag{18}$$

We use Eq. (18), along with Eq. (14), to predict $\langle n_i \rangle$. We perform simulations following the same procedure described in the previous section, using $m_i = k_i$, for both the Erdös-Rényi and the Barabási-Álbert networks. The results are plotted in Figs. 5 and 6. We see that again the theory predicts very well the results of the simulations.

VI. CONCLUSIONS AND REMARKS

We proposed and analyzed a model of particles randomly moving in a network, inspired by the motion of information packets in the Internet. The limited number of particles the nodes can support introduces an effective interaction among the particles, which makes this a challenging problem in statistical physics. The problem is further complicated by the distinguishable nature of the particles. By using a mathematical trick, we were able to sum the grand-canonical partition function and thus derive analytical expressions for the average occupation number of each node. We have tested the theory by comparing the predicted results with simulations, and they have been confirmed to high accuracy.

We have seen that highly connected nodes are populated preferentially, as the particle density increases. For even moderate densities, highly connected nodes rapidly become maximally occupied, which means that they contribute to

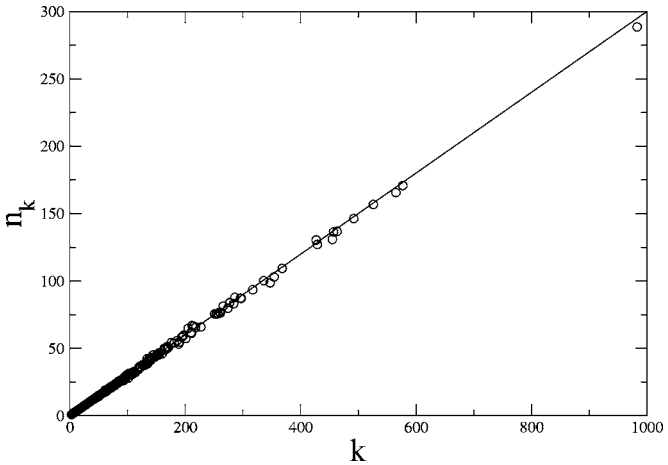

FIG. 6. Mean occupation n_k of nodes with degree k, with each node i supporting at most k_i particles, in a Barabási-Álbert network with the same parameters as in Fig. 2. The number of particles is 30% of the maximum number supported by the network (which is the sum of the degrees of all nodes). Circles are values measured from the simulation, and the continuous line is the prediction of Eq. (18).

choking traffic from their neighbors, since particles cannot move through nodes with maximum occupation. Many real networks have a scale-free topology, which implies that there are a few very highly connected nodes (hubs). These hubs are fundamental to the efficient transport in the network, and when they are blocked due to maximum occupation, transport is seriously disrupted. Consequently, in scale-free networks congestion tends to happen purely as a result of their topology. The way to deal with this is to distribute the resources among the nodes inhomogeneously, so that highly connected nodes can support more particles. As seen in Fig. 6 of Sec. V, even for relatively high particle densities the highly connected nodes are occupied below their capacity, allowing traffic to flow freely.

As a last remark, we note that if all nodes can support at most one particle, we recover the Fermi-Dirac distribution, as shown in Sec. III. One may find this strange, since we know from quantum mechanics that fermions (for example, electrons) are indistinguishable, whereas in the derivation of Eq. (13) we assumed that the particles were distinguishable. But there is no contradiction here. It turns out that for the sole case of $m_i=m=1$, the statistics predicted by our theory is the same as that predicted by the Gentile distribution, which assumes indistinguishability (and of course reduces to the Fermi-Dirac distribution in this case). In other words, distinguishable and indistinguishable particles in this particular case have the same statistics. This is a particular feature of $m=1$, and it is not true for any $m>1$, as we have seen in Sec. IV.

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